Lieb–Robinson Bounds and Entanglement Limits in Quantum Dynamics with Finite-Dimensional Memory

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April 26, 2026

Abstract

We derive an extension of the Lieb–Robinson bound that accounts for memory effects in quantum lattice systems evolving under discrete-time non-Markovian dynamics generated by local interactions with finite-dimensional memory subsystems associated with each bond. The global evolution is unitary on an extended Hilbert space

$$\mathcal{H}_{\text{ext}} = \bigotimes_{i=1}^{N} \mathcal{H}_i \otimes \bigotimes_{bonds} \mathcal{H}_M^{(i)},$$

while non-Markovian behavior, arising from repeated coupling to the same bond memory at successive time steps, leads to temporal correlations in the system.

In this work, "finite memory" refers specifically to memory subsystems of fixed finite Hilbert space dimension whose reduced dynamics satisfy an exponential mixing condition in operator norm, meaning that correlation functions decay exponentially over time, with a characteristic timescale τ*. More precisely, the memory channels satisfy*

$$\|\mathcal{T}_M^s(X) - \text{Tr}(X)\,\sigma_M\| \le c\,e^{-s/\tau}\,\|X\|,$$

for all memory observables X*, where* σ_M *is a stationary state, and* $c > 0$ *depends only on the memory dynamics.*

For a one-dimensional lattice with nearest-neighbor interactions, we prove that the connected correlation function, defined as

$$C_{i,j}(t) := \sup_{\substack{A_i = A_i^\dagger,\, B_j = B_j^\dagger \\ \|A_i\| \le 1,\, \|B_j\| \le 1}} \left|\text{Tr}[(A_i \otimes B_j)\rho(t)] - \text{Tr}[A_i \rho_i(t)]\ \text{Tr}[B_j \rho_j(t)]\right|,$$

satisfies an exponential bound of the form

$$C_{i,j}(t) \le C_0 \exp\Big[-\frac{d(i,j) - v_{\text{eff}}(\tau)\,t}{\xi}\Big],$$

where $d(i,j) = |i - j|$ *and* $\xi > 0$ *is independent of system size and time. The effective velocity* $v_{\text{eff}}(\tau)$ *is explicitly bounded in terms of the local interaction strength and the memory mixing parameters, ensuring the system exhibits a finite light cone, meaning that information propagation is confined to a region growing linearly with time, with a well-defined maximum speed.*

In the limit $\tau \to \infty$*, the bound no longer guarantees a finite propagation speed, reflecting the transition to a persistent memory regime where correlations do not decay exponentially.*

As an operational consequence, the entanglement fidelity between two-site reduced states $\rho_{i,j}(t)$ *and a maximally entangled state satisfies*

$$F_{i,j}(t) = \langle \Phi^+ | \rho_{i,j}(t) | \Phi^+ \rangle \le \tfrac{1}{2}\big[1 + C_{i,j}(t)\big].$$

These results extend Lieb–Robinson theory to a class of discrete-time quantum dynamics with memory subsystems associated with each bond, enabling resummation of multi-step memory effects into an effective correlation propagation kernel that governs the decay of temporal correlations. This establishes a quantitative relationship between the timescale of memory mixing and the effective speed of information propagation in the system.

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1 Introduction

Understanding the propagation of correlations and entanglement in quantum many-body systems is central to quantum information processing, communication, and simulation. A key structural result in this context is the Lieb–Robinson bound, which establishes that correlations in locally interacting quantum systems propagate with a finite effective velocity, giving rise to a light-cone structure even in non-relativistic settings. While these bounds are well understood for closed systems and certain classes of Markovian open dynamics, the behavior of correlation propagation in systems with **temporally correlated noise** remains less fully characterized.

1.1 System and Bond-Wise Memory Structure

We consider a one-dimensional lattice of N finite-dimensional quantum systems, with site Hilbert spaces $\mathcal{H}_i \simeq \mathbb{C}^d$. We equip the lattice with the standard graph distance

$$d(i,j) := |i-j|.$$

The system Hilbert space is

$$\mathcal{H}_{\text{sys}} = \bigotimes_{i=1}^{N} \mathcal{H}_i.$$

To model finite-memory non-Markovian effects, we introduce **bond-wise memory subsystems** $\mathcal{H}_M^{(i)}$ for each nearest-neighbor bond $i = 1, \ldots, N-1$. Each memory subsystem is finite-dimensional and mediates temporal correlations between sites i and $i+1$. The total memory space is

$$\mathcal{H}_M = \bigotimes_{i=1}^{N-1} \mathcal{H}_M^{(i)}.$$

The **extended Hilbert space** including all memory subsystems and ancillary degrees of freedom is

$$\mathcal{H}_{\text{ext}} = \mathcal{H}_{\text{sys}} \otimes \mathcal{H}_M \otimes \mathcal{H}_E,$$

where $\mathcal{H}_E$ denotes the global ancilla space (countable tensor product over time steps and bonds), used in the collision-model dynamics.

1.2 Collision-Model Dynamics

The system evolves in discrete time steps $t = 0, 1, 2, \ldots$ via a **layered circuit of local unitaries** $U_i(t)$ acting on site pairs $(i, i+1)$, their corresponding memory $\mathcal{H}_M^{(i)}$, and fresh ancillas $\mathcal{H}_{E,t}^{(i)}$. Within each even/odd layer, unitaries on disjoint bonds commute, ensuring strict locality:

$$U(t) = U^{(\text{even})}(t)\, U^{(\text{odd})}(t), \quad U^{(\text{odd/even})}(t) = \prod_{i \in \text{odd/even}} U_i(t).$$

The reduced system state is obtained by tracing out all memory and ancilla degrees of freedom:

$$\rho(t) = \mathrm{Tr}_{M,E}\big[\rho_{\mathrm{ext}}(t)\big].$$

1.3 Finite Memory and Mixing

Each bond-wise memory subsystem $\mathcal{H}_M^{(i)}$ satisfies an **exponential mixing condition**. To formalize this, we introduce a reference memory channel

$$\mathcal{T}_M^{(i)}(X) := \mathrm{Tr}_{i,i+1}\big[U_i(\rho^*_{\mathrm{sys}} \otimes X \otimes \eta_E)U_i^\dagger\big],$$

where ρ^*_{sys} is a fixed, full-rank reference state on sites $(i, i+1)$. *Crucially, $\mathcal{T}_M^{(i)}$ is* not *the physical reduced dynamics of the memory subsystem; rather, it is a technical auxiliary object used to establish exponential decay bounds for memory correlations.*

In particular, exponential contraction of the reference channel $\mathcal{T}_M^{(i)}$ guarantees that the actual reduced memory state $\rho_M^{(i)}(t)$ also converges exponentially to a stationary state, thereby connecting the channel-based formulation here to the state-based mixing bounds used in Section 2.

There exist constants $c > 0$, $\tau > 0$, and stationary states $\sigma_M^{(i)}$ such that

$$\|(\mathcal{T}_M^{(i)})^s(X) - \mathrm{Tr}(X)\,\sigma_M^{(i)}\| \le c\,e^{-s/\tau}\|X\|.$$

This ensures that temporal correlations mediated by each bond decay exponentially, providing a finite-memory timescale τ.

Reference system state. The state ρ^*_{sys} is a fixed, full-rank reference state on $\mathcal{H}_i \otimes \mathcal{H}_{i+1}$ used only in the definition of the memory transfer channel. It is not time-evolving and does not depend on the initial condition. Its role is to define a consistent contraction reference for the memory dynamics and does not affect the physical system evolution.

Uniform memory channel assumption. For simplicity of notation, we assume that all bond-wise memory channels are identical up to translation invariance, i.e.

$$\mathcal{T}_M^{(i)} \equiv \mathcal{T}_M \quad \forall i \in \{1, \ldots, N-1\}.$$

Accordingly, we drop the superscript (i) when no ambiguity arises.

1.4 Locality and Correlation Propagation

The layered circuit enforces that information spreads **at most one site per time step**, giving rise to an emergent light-cone structure. The connected correlation function between sites i and j is defined as

$$C_{i,j}(t) := \sup_{\substack{A_i = A_i^\dagger,\, B_j = B_j^\dagger \\ \|A_i\| \le 1,\, \|B_j\| \le 1}} \left|\mathrm{Tr}[(A_i \otimes B_j)\rho(t)] - \mathrm{Tr}[A_i\rho_i(t)]\ \mathrm{Tr}[B_j\rho_j(t)]\right|,$$

where $\rho_i(t) := \mathrm{Tr}_{\{k\neq i\}}[\rho(t)]$ and $\rho_j(t) := \mathrm{Tr}_{\{k\neq j\}}[\rho(t)]$ denote the reduced single-site states, and $A_i(t) := U(0,t)^\dagger A_i\, U(0,t)$ denotes the Heisenberg evolution of a local observable A_i under the full extended dynamics, where $U(0,t))$ is the propagator defined in Section 2, which governs the evolution of local observables.

1.5 Main Result (Informal Statement)

Under the above bond-wise memory and locality assumptions, the connected correlations satisfy a **Lieb–Robinson-type bound** of the form:

$$C_{i,j}(t) \le C_0 \exp\left[-\frac{d(i,j) - v_{\text{eff}}(\tau)\, t}{\xi}\right],$$

where $C_0 > 0$ and $\xi > 0$ are constants depending only on local Hilbert space dimension and interaction parameters (but independent of system size and time); and with effective velocity

$$v_{\text{eff}}(\tau) = v_0 \left(1 + \frac{c}{1 - e^{-1/\tau}}\right),$$

where $v_0 = \mathcal{O}(J_0)$ is the standard memoryless nearest-neighbor velocity, and c depends on the memory mixing constants. Finite τ guarantees a finite light-cone, while $\tau \to \infty$ corresponds to fully persistent memory, for which no finite velocity can be guaranteed.

1.6 Operational Consequences

The bound provides direct limits on **entanglement distribution**: for two-site reduced states $\rho_{i,j}(t)$,

$$F_{i,j}(t) = \langle \Phi^+ | \rho_{i,j}(t) | \Phi^+ \rangle \le \tfrac{1}{2}[1 + C_{i,j}(t)], \quad \text{where } |\Phi^+\rangle := \tfrac{1}{\sqrt{d}} \sum_{k=1}^{d} |k\rangle \otimes |k\rangle.$$

1.7 Relation to Prior Work

Lieb–Robinson bounds were originally established for locally interacting quantum spin systems with Hamiltonian dynamics [9], and later extended to time-dependent Hamiltonians and more general interaction structures [11].

Extensions to open-system dynamics include dissipative Lindblad evolutions and Markovian quantum channels, where Lieb–Robinson-type bounds have been derived using operator-norm and generator techniques [12, 8, 1]. These results establish finite-velocity propagation in the absence of memory effects and form the basis for locality in Markovian quantum dynamics.

More recent work has explored long-range interacting systems, where the effective light-cone structure can be modified but remains sharply controlled under suitable decay conditions [6].

Non-Markovian quantum dynamics has been studied extensively in terms of memory kernels and information backflow, including process-tensor and master-equation approaches [13]. However, these frameworks typically characterize temporal correlations rather than providing explicit spatial Lieb–Robinson-type velocity bounds in extended lattice systems.

Collision models provide a microscopic framework for non-Markovian dynamics via repeated system–environment interactions [10, 4]. In contrast to previous approaches, the present work establishes a rigorous Lieb–Robinson-type bound for a spatially extended lattice system with bond-wise persistent memory, where temporal memory effects are incorporated into an effective spatial propagation kernel while preserving a finite light-cone structure.

1.8 Scope and Organization

We consider one-dimensional lattices with nearest-neighbor interactions, discrete-time unitary collision dynamics, finite-dimensional local and bond-wise memory Hilbert spaces, and exponentially mixing finite-memory subsystems. Section 2 introduces the collision-model framework and memory structure, Section 3 defines the connected correlation measures, Sections 4–5 develop the effective kernel and correlation propagation bounds, and Section 6 provides the complete proof of the memory-dependent Lieb–Robinson theorem.

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2 State Evolution and Model

2.1 System and extended Hilbert space

We consider a one-dimensional quantum lattice with sites

$$V := \{1, 2, \ldots, N\},$$

and local Hilbert spaces $\mathcal{H}_i$ of finite dimension. We equip the lattice V with the standard graph distance

$$d(i, j) := |i - j|.$$

The system Hilbert space is

$$\mathcal{H}_{\text{sys}} = \bigotimes_{i=1}^{N} \mathcal{H}_i.$$

To model non-Markovian dynamics with finite memory, we introduce a local memory subsystem associated with each nearest-neighbor bond. The total memory Hilbert space is

$$\mathcal{H}_M = \bigotimes_{i=1}^{N-1} \mathcal{H}_M^{(i)},$$

where each $\mathcal{H}_M^{(i)}$ has uniformly bounded finite dimension independent of system size N and stores information mediating interactions between sites i and $i+1$ across time steps.

In addition, for each bond i and time step $t \in \mathbb{N}$, we introduce an environment ancilla space $\mathcal{H}_{E,t}^{(i)}$, all of identical finite dimension. The environment is defined inductively in time. At each time step t, only ancillas $\{\mathcal{H}_{E,s}^{(i)} : s \leq t,\ i = 1, \ldots, N-1\}$ are included in the dynamical description. The full infinite tensor product is understood as a formal bookkeeping device representing the inductive limit of these finite-time Hilbert spaces.

Thus, at finite time t, the evolution is always defined on a finite-dimensional Hilbert space.

We denote the full extended Hilbert space at time t by

$$\mathcal{H}_{\text{ext}}(t) := \mathcal{H}_{\text{sys}} \otimes \mathcal{H}_M \otimes \bigotimes_{s=0}^{t} \bigotimes_{i=1}^{N-1} \mathcal{H}_{E,s}^{(i)}.$$

When no ambiguity arises, we suppress the time dependence and write $\mathcal{H}_{\text{ext}}$.

2.2 Collision-model dynamics

The evolution proceeds in discrete time steps $t = 0, 1, 2, \ldots$. At each time step, the system interacts locally with memory and fresh ancillas via a layered unitary circuit.

For each bond i, define a unitary operator

$$U_i(t) : \mathcal{H}_i \otimes \mathcal{H}_{i+1} \otimes \mathcal{H}_M^{(i)} \otimes \mathcal{H}_{E,t}^{(i)} \longrightarrow \mathcal{H}_i \otimes \mathcal{H}_{i+1} \otimes \mathcal{H}_M^{(i)} \otimes \mathcal{H}_{E,t}^{(i)}.$$

We assume a uniform bound on the interaction strength: there exists a constant $J_0 > 0$ such that all locality estimates derived from $U_i(t)$ are controlled by J_0, uniformly in i and t.

Each memory subsystem $\mathcal{H}_M^{(i)}$ is treated as a physical quantum degree of freedom in the global Hilbert space. It is not modeled as an abstract channel memory or an effective classical register, but as part of the unitary-evolving closed system defined on $\mathcal{H}_{\text{ext}}$.

We assume that these unitaries are uniformly bounded and identical up to translation (or satisfy uniform norm bounds).

To ensure locality, we implement a two-layer (even/odd) decomposition:

$$U^{(\text{odd})}(t) = \prod_{\substack{i=1 \\ i \text{ odd}}}^{N-1} U_i(t), \quad U^{(\text{even})}(t) = \prod_{\substack{i=1 \\ i \text{ even}}}^{N-1} U_i(t),$$

and define the full time-step unitary as

$$U(t) = U^{(\text{even})}(t)\, U^{(\text{odd})}(t).$$

Within each layer (odd or even), the unitaries $U_i(t)$ act on disjoint tensor factors and therefore commute exactly. As a result, the ordering of the product within each layer is irrelevant.

Locality structure. Each unitary $U_i(t)$ acts only on:

- sites i and $i+1$,
- the corresponding memory subsystem $\mathcal{H}_M^{(i)}$,
- the fresh ancilla $\mathcal{H}_{E,t}^{(i)}$.

Each ancilla $\mathcal{H}_{E,t}^{(i)}$ is initialized in a fixed product state $\eta_E^{(i)}$, independent of both t and i, and is assumed to interact with the system exactly once at time step t. After interaction, it is never reused.

No operation couples distant sites directly, and no memory subsystem is shared between different bonds.

2.3 Global unitary evolution

Define the time-ordered propagator

$$U(0,t) := U(t-1)\, U(t-2)\, \cdots\, U(0).$$

The extended system state evolves unitarily as

$$\rho_{\text{ext}}(t) = U(0,t)\, \rho_{\text{ext}}(0)\, U(0,t)^{\dagger}.$$

The reduced system state is obtained by tracing out memory and environment:

$$\rho(t) = \operatorname{Tr}_{M,E}\big[\rho_{\text{ext}}(t)\big].$$

No partial trace is performed at intermediate times in the definition of the global unitary evolution. All reductions to system or memory observables are performed only after the full unitary evolution up to time t. The memory subsystems and ancillas remain part of the full unitary dynamics until the final time t, ensuring that all non-Markovian effects arise solely from unitary system–memory–environment interactions.

This reduced dynamics is, in general, non-Markovian due to the persistent memory subsystems.

2.4 Initial conditions

We assume a product initial state of the form

$$\rho_{\text{ext}}(0) = \rho_{\text{sys}}(0) \otimes \rho_M(0) \otimes \rho_E,$$

where:

- $\rho_{\text{sys}}(0) = \bigotimes_{i=1}^{N} \rho_i(0)$,
- $\rho_M(0) = \bigotimes_{i=1}^{N-1} \rho_M^{(i)}(0)$,
- $\rho_E = \bigotimes_{t,i} \eta_E^{(i)}$ is a product of identical ancilla states.

This assumption ensures that all correlations are generated dynamically.

2.5 Memory persistence and mixing

The reduced memory dynamics is defined by the restriction of the global unitary evolution followed by tracing out system and environment degrees of freedom. The reduced memory dynamics is therefore defined only as a derived object from the full unitary evolution; it is not generated by intermediate coarse-graining during the time evolution. This induces a (generally non-Markovian) family of memory states $\rho_M^{(i)}(t)$.

This induced reduced evolution is not assumed to factor into a single-step completely positive map on system or memory alone. Instead, all bounds involving memory are derived directly from the global unitary dynamics combined with the exponential mixing property.

To quantify finite memory effects, we assume exponential convergence in time of the reduced memory states.

All norms on $\mathcal{H}_M^{(i)}$ are equivalent since the space is finite-dimensional, and all constants may be adjusted accordingly without changing scaling in t.

Let $\rho_M^{(i)}(t)$ denote the reduced state of the memory subsystem $\mathcal{H}_M^{(i)}$ at time t. We assume that there exist constants $c > 0$, $\tau > 0$, and a reference state $\sigma_M^{(i)}$ such that for all $s \geq 0$,

$$\left\| \rho_M^{(i)}(t+s) - \sigma_M^{(i)} \right\|_1 \leq c\, e^{-s/\tau}.$$

This condition expresses that memory correlations decay exponentially over time, with characteristic timescale τ.

In particular, trace-norm decay implies corresponding decay in operator norm up to a fixed constant factor absorbed into c.

Time-homogeneity assumption. We assume that the memory mixing bound holds uniformly in time, i.e. the constants c and τ do not depend on the initial time t. This is sufficient for all subsequent uniform estimates.

Memory-history contraction lemma. The exponential mixing condition implies a uniform bound on all multi-step memory contributions.

For each bond i, the reduced memory evolution is governed by the channel $\mathcal{T}_M^{(i)}$ defined as the effective reduced dynamics induced by the bond unitary $U_i(t)$ after tracing out system and ancilla degrees of freedom. Since all bonds satisfy identical bounds, we suppress the index i in estimates and write $\mathcal{T}_M$ for a generic representative channel.

Under the time-homogeneity assumption, this channel acts uniformly across all bonds and time steps.

Lemma 1 (Collapse of memory histories)**.** *Let $\|\cdot\|$ denote the operator norm on the finite-dimensional memory space. By duality between trace norm and operator norm in finite dimensions, together with norm equivalence on $\mathcal{H}_M^{(i)}$, we obtain*

$$\|(\mathcal{T}_M)^s(X) - \mathrm{Tr}(X)\sigma_M\| \leq c\, e^{-s/\tau}\|X\|.$$

All constants are understood up to finite dimension-dependent rescaling absorbed into c.

Then for any sequence of memory interactions over time steps $s = 0, \ldots, t$, the total contribution satisfies

$$\sum_{s=0}^{t} \|(\mathcal{T}_M)^s(X_s)\| \leq \frac{c}{1 - e^{-1/\tau}} \sup_{0\leq s\leq t} \|X_s\|.$$

where X_s are bounded operators on the memory Hilbert space $\mathcal{H}_M^{(i)}$ (suppressing the index i).

In particular, all memory-dependent histories can be replaced by a single effective bounded contribution depending only on τ, independent of t.

2.6 Heisenberg evolution

For any system observable $A \in \mathcal{B}(\mathcal{H}_{\mathrm{sys}})$, its Heisenberg evolution is defined on the extended space as

$$A(t) := U(0,t)^\dagger\, A\, U(0,t).$$

Although $A(t)$ acts on the full extended Hilbert space, we define its spatial support as the subset of lattice sites on which it acts nontrivially in the system degrees of freedom.

System restriction of observables. For an observable $A(t)$ acting on $\mathcal{H}_{\mathrm{ext}}$, we define its effective system observable via partial trace duality with respect to the fixed reference state on memory and ancillas:

$$A_{\mathrm{sys}}(t) := \mathrm{Tr}_{M,E}\big[A(t)\,(\mathbb{I}_{\mathrm{sys}} \otimes \rho_M(0) \otimes \rho_E)\big].$$

This corresponds to evaluating $A(t)$ on memory and ancilla degrees of freedom initialized in their reference states, effectively treating them as auxiliary subsystems.

The support of $A_{\mathrm{sys}}(t)$ is understood as the minimal subset of system sites required to represent it as an operator in $\mathcal{B}(\mathcal{H}_{\mathrm{sys}})$.

2.7 Finite-speed propagation of support

By construction, each time step enlarges the system support of observables by at most one lattice spacing. Therefore, for any observable initially supported on site i,

$$\mathrm{supp}(A_i(t)) \subseteq \{x \in V : d(x,i) \leq t\}.$$

This locality property is a direct consequence of the layered nearest-neighbor unitary structure and forms the basis for Lieb–Robinson-type bounds derived in subsequent sections.

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3 Correlation Measures

3.1 Connected Correlation Function

Throughout this section, the primary formulation is in the Schrödinger picture in terms of the state $\rho(t)$. Any use of Heisenberg-evolved observables such as $A_i(t)$ is purely notational and refers to the definition in Section 2. These expressions are used only for comparison with standard Lieb–Robinson derivations and do not change the underlying definition of the correlation function.

All expectation values are taken with respect to the reduced system state

$$\rho(t) := \mathrm{Tr}_{M,E}[\rho_{\mathrm{ext}}(t)],$$

as defined in Section 2.

To quantify correlations between sites i and j, we define the connected correlation function

$$C_{i,j}(t) := \sup_{\substack{A_i=A_i^\dagger,\, B_j=B_j^\dagger \\ \|A_i\|\le 1,\, \|B_j\|\le 1}} \left| \mathrm{Tr}[(A_i \otimes B_j)\rho(t)] - \mathrm{Tr}[A_i\rho_i(t)]\ \mathrm{Tr}[B_j\rho_j(t)] \right|,$$

where:

- $A_i \in \mathcal{B}(\mathcal{H}_i)$ and $B_j \in \mathcal{B}(\mathcal{H}_j)$ are Hermitian observables with operator norm bounded by 1,
- $\rho_i(t) := \mathrm{Tr}_{\{k\neq i\}}[\rho(t)]$ and $\rho_j(t)$ are reduced density matrices of the system state $\rho(t)$.

This quantity depends on the state $\rho(t)$ generated by the dynamics defined in Section 2, and is evaluated for arbitrary initial product states specified in Section 2.

Although the correlation function $C_{i,j}(t)$ is defined entirely in the Schrödinger picture in terms of the state $\rho(t)$, we will occasionally refer to the Heisenberg-evolved observable $A_i(t)$ (defined in Section 2) as a notational convenience when relating the present formulation to standard Lieb–Robinson derivations. This use is purely auxiliary and not part of the definition of $C_{i,j}(t)$.

3.1.1 Interpretation

The function $C_{i,j}(t)$ satisfies:

- $C_{i,j}(t) = 0$ if $\rho(t)$ factorizes between sites i and j,
- $C_{i,j}(t)$ increases as correlations (classical or quantum) build up under the dynamics,
- $C_{i,j}(t)$ provides a distance-resolved measure of correlation spreading across the lattice.

Importantly, no reference to Heisenberg-picture evolution or commutator expressions is used in this definition. The quantity is entirely defined in the Schrödinger picture.

3.2 Basic Properties

By standard operator norm and trace inequalities in finite-dimensional Hilbert spaces, one has

$$0 \le C_{i,j}(t) \le 1.$$

The lower bound follows from positivity of the absolute value, and the upper bound follows from:

- The inequality $|\mathrm{Tr}(A\rho)| \le \|A\|\,\|\rho\|_1$ for trace-class operators,
- the fact that $\rho(t)$ is a density operator satisfying $\|\rho(t)\|_1 = 1$,
- the operator norm bounds $\|A_i\| \le 1$, $\|B_j\| \le 1$.

Thus, $C_{i,j}(t)$ is a normalized measure of operator-based correlations between sites i and j.

3.3 Emergent Effective Correlation Kernel

This subsection derives the effective correlation propagation kernel used in Sections 4 and 5 without introducing any Markov approximation at the level of the reduced system dynamics. All effective kernels and coefficients introduced in this subsection are used solely as tools for bounding the growth of correlations. They do not represent an exact reduced dynamics of the system, and no Markovian closure is assumed at the level of the system state.

Single-step correlation propagation mechanism. The key step in deriving the correlation recursion is to relate the evolution of connected correlations at time $t+1$ to those at time t under the collision-model dynamics.

We formalize this as follows.

Lemma 2 (One-step correlation propagation bound)**.** *Let the dynamics be the layered unitary collision model defined in Section 2. Then there exists a family of non-negative coefficients $K_{i,k}(\tau)$, defined as uniform operator-norm bounds induced by the Heisenberg evolution $A_i(t) = U(0,t)^\dagger A_i U(0,t)$ (see Section 2), such that for all sites $i, j \in V$,*

$$C_{i,j}(t+1) \le \sum_{k:\,|i-k|\le 1} K_{i,k}(\tau)\, C_{k,j}(t).$$

More precisely, $K_{i,k}(\tau)$ is obtained as the supremum over all admissible local observables A_i, B_j in the definition of $C_{i,j}(t)$ of the contributions to $A_i(t+1)$ supported on site k, after tracing out memory and ancilla degrees of freedom as specified in Section 2.

$K_{i,k}(\tau)$ depends only on: (i) upper bounds of the local interaction strength J_0, (ii) the memory mixing parameters c, τ, and (iii) finite-dimensionality of the memory subsystem, but not on system size or time.

The coefficients $K_{i,k}(\tau)$ are not uniquely defined; they represent uniform upper bounds obtained by maximizing over all allowed microscopic realizations consistent with the unitary dynamics and memory mixing condition.

Proof sketch. The proof proceeds in three steps:

(i) Locality of unitary evolution. Since each time-step unitary $U(t)$ is a product of nearest-neighbor gates, the Heisenberg evolution of any local observable A_i satisfies

$$A_i(t+1) = U^\dagger(t)\, A_i(t)\, U(t),$$

and its support can only extend to sites $i-1, i, i+1$ and their associated memory subsystems, including memory degrees of freedom, which are strictly localized to bonds.

(ii) Decomposition into system and memory contributions. After tracing out memory and ancilla degrees of freedom, the evolution of observables can be bounded using operator norm inequalities derived from the unitary dynamics, yielding a decomposition into local propagation terms and memory-mediated contributions. This yields an intermediate upper bound, obtained by estimating each local contribution separately, of the form

$$C_{i,j}(t+1) \le \sum_{k:\,|i-k|\le 1} e_{i,k}(t)\, C_{k,j}(t),$$

where $e_{i,k}(t) \ge 0$ are scalar coefficients obtained as operator-norm bounds on the contributions to correlation propagation induced by the unitary dynamics after tracing out memory and ancillas. These coefficients arise from taking supremum bounds over all admissible local observables in the definition of $C_{i,j}(t)$, ensuring that they provide uniform estimates independent of the specific choice of observables. No completely positive or Markovian structure is assumed.

This bound is obtained by decomposing the evolved observable $A_i(t+1)$ into a finite sum of contributions supported on neighboring sites and estimating each contribution separately. The definition of $C_{i,j}(t)$ as a supremum over local observables allows these contributions to be combined linearly at the level of upper bounds.

(iii) Memory-history contraction. Using the memory-history contraction lemma, all contributions arising from repeated interactions with the memory subsystem over multiple time steps form a geometrically suppressed series.

Consequently, the full history-dependent evolution induced by the memory can be bounded by a single effective contribution:

$$e_{i,k}(t) \le K_{i,k}(\tau),$$

where $K_{i,k}(\tau)$ absorbs the convergent geometric series of memory-mediated corrections and depends only on (c, τ). These bounds are obtained by applying the memory-history contraction lemma to the cumulative contribution of memory-mediated terms.

Combining (i)–(iii) yields the stated inequality. □

Consequence. This lemma provides the rigorous justification for the correlation recursion used in Section 4:

$$C(t+1) \le K(\tau)\, C(t),$$

where $K(\tau)$ is the effective correlation kernel defined in the following sections.

No Markovian closure at the level of states. No assumption of Markovian dynamics is made for the reduced system state $\rho(t)$, which remains generally non-Markovian due to persistent memory degrees of freedom.

However, by using the exponential mixing property of the memory subsystems, we derive an *effective one-step upper bound* for correlation propagation:

$$C(t+1) \le K(\tau)\, C(t),$$

which serves as a comparison inequality. Here $K(\tau)$ denotes the matrix with entries $K_{i,k}(\tau)$ acting on the vector of correlations.

Interpretation. The kernel $K(\tau)$ encodes two effects: (i) nearest-neighbor spatial propagation governed by J_0, (ii) temporally suppressed memory feedback encoded in the finite-time mixing scale τ.

The object $K(\tau)$ should be interpreted as a bookkeeping device for upper bounds on correlation propagation derived from the unitary dynamics and the memory mixing assumption. It is not introduced as a fundamental dynamical generator.

It provides a controlled upper-bound propagation mechanism that reflects both locality and finite-memory decay, without implying an exact Markovian description of the underlying process.

3.4 Role in the Present Work

The correlation function $C_{i,j}(t)$ will be shown in Section 4 to satisfy a closed recursive inequality directly derived from the collision-model dynamics with finite memory. This recursion forms the basis of the Lieb–Robinson-type bound established in this work.

Lemma 3 (Relation between commutator and state correlations)**.** *Let A_i, B_j be local observables on the system Hilbert space, and let $A_i(t)$ denote their Heisenberg evolution defined on the extended space. Then for any state $\rho(t)$,*

$$|\mathrm{Tr}(\rho(t)[A_i(t), B_j])| \leq 4C_{i,j}(t).$$

Taking the supremum over all trace-class operators ρ with $\|\rho\|_1 \leq 1$ yields the corresponding operator norm bound

$$\|[A_i(t), B_j]\| \leq 4C_{i,j}(t).$$

Proof sketch. Expanding the commutator expectation value in state $\rho(t)$ and using standard operator norm inequalities yields

$$|\mathrm{Tr}(\rho[A, B])| \leq 2\left(|\mathrm{Tr}(\rho AB)| + |\mathrm{Tr}(\rho BA)|\right).$$

Subtracting and adding factorized expectations and applying the definition of $C_{i,j}(t)$ gives the result.

Thus, the commutator-based Lieb–Robinson quantity is controlled by the state-based correlation function used throughout this work, up to a constant factor.

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4 Correlation Propagation with Memory Effects

4.1 Setup and Notation

We consider the one-dimensional lattice $V = \{1, \ldots, N\}$ with system Hilbert spaces $\mathcal{H}_i$ of finite dimension and the global system space $\mathcal{H}_{\mathrm{sys}} = \bigotimes_{i=1}^{N} \mathcal{H}_i$ as defined in Section 2.

Let $C_{i,j}(t)$ denote the connected correlation function of sites $i, j \in V$ at time t, defined in Section 3. For a fixed reference site $j \in V$, define the *correlation vector*

$$C(t) := \left(C_{i,j}(t)\right)_{i \in V} \in \mathbb{R}_{\geq 0}^{|V|}.$$

We aim to rigorously bound the propagation of correlations under the collision-model dynamics, taking into account both local unitary interactions and memory-assisted feedback.

Role of the effective kernel $K(\tau)$. Recall that the coefficients $K_{i,k}(\tau)$ were introduced in Section 3 as uniform upper bounds on one-step correlation propagation derived from the unitary collision-model dynamics. They are not associated with any reduced dynamical map on the system.

The dependence on the memory timescale τ arises through the memory-history contraction mechanism of Section 2: repeated interactions with the same memory subsystem generate contributions at different time depths, which are exponentially suppressed and summable. The coefficients $K_{i,k}(\tau)$ are defined so as to absorb this entire resummed contribution into a single effective nearest-neighbor bound.

Thus, $K(\tau)$ should be interpreted throughout as a comparison operator controlling correlation growth, incorporating both locality of the unitary dynamics and the finite-memory decay encoded by τ.

4.2 One-Step Correlation Propagation Bound

We begin by formalizing the local propagation of correlations over a single time step.

Lemma 4 (One-step correlation propagation)**.** *Let $C_{i,j}(t)$ be the connected correlation function at time t. Then, there exist non-negative coefficients $K_{i,k}(\tau)$ such that*

$$C_{i,j}(t+1) \leq \sum_{k:|i-k|\leq 1} K_{i,k}(\tau)\, C_{k,j}(t),$$

for all $i \in V$, where:

1. *$|i-k| \leq 1$ enforces nearest-neighbor locality of each unitary $U_i(t)$,*
2. *$K_{i,k}(\tau)$ depends only on the local interaction strength J_0, the memory mixing parameters c, τ, and the finite dimension of $\mathcal{H}_M^{(i)}$,*
3. *The coefficients absorb all memory-mediated contributions via the memory-history contraction lemma (Section 2.7).*

Proof Sketch. 1. **Locality:** By construction of $U(t)$ as a layered nearest-neighbor unitary circuit (Section 2.3–2.4), any Heisenberg-evolved observable $A_i(t+1) = U(t)^\dagger A_i(t) U(t)$ has support only on sites $i-1, i, i+1$ and the associated bond memories.

2. **Partial trace over memory and ancillas:** Using the system restriction definition from Section 2.9, one can bound contributions to $A_i(t+1)$ on each site k by operator norm inequalities.

Clarification on cumulative memory feedback. The memory-history contraction lemma (Section 2.7) applies not only to a single application of the memory channel, but to the entire *sequence of repeated interactions* between system and memory across all intermediate time steps.

Concretely, a contribution to $C_{i,j}(t+1)$ arising from memory interactions at depth s corresponds to s successive applications of the reduced memory transfer mechanism. These contributions form a nested sequence of memory-mediated corrections, which accumulate additively at the level of operator-norm bounds.

By exponential mixing, this multi-step contribution forms a geometrically decaying series in s, ensuring that the total cumulative effect of all past memory feedback up to time t remains uniformly bounded by a constant of order

$$\sum_{s=0}^{t} c e^{-s/\tau} \leq \frac{c}{1-e^{-1/\tau}}.$$

As a result, the coefficients $K_{i,k}(\tau)$ absorb the entire history of memory-mediated corrections uniformly in time, and do not grow with t.

3. **Memory-history contraction:** By Lemma 1, contributions from repeated interactions with memory form a geometrically suppressed series $\sum_{s=0}^{\infty} ce^{-s/\tau} \leq c/(1 - e^{-1/\tau})$. Including the base interaction scale J_0 gives uniform coefficients $K_{i,k}(\tau)$.

Combining these steps yields the stated inequality. □

4.3 Definition of the Effective Correlation Kernel

We define the *effective correlation kernel* $K(\tau) \in \mathbb{R}_{\geq 0}^{|V|\times|V|}$ with entries

$$K_{i,k}(\tau) := \begin{cases} \text{the bound from Lemma 4}, & |i-k| \leq 1, \\ 0, & |i-k| > 1. \end{cases}$$

Physical interpretation of $K(\tau)$. The kernel $K(\tau)$ can be understood as a *one-step information propagation envelope* for the dynamics.

Its matrix elements $K_{i,k}(\tau)$ quantify the maximal contribution of correlations initially localized at site k to correlations at site i after a single time step, after fully accounting for:

(i) direct nearest-neighbor spreading generated by the unitary gates, and (ii) indirect memory-mediated feedback accumulated over past interactions.

In this sense, $K(\tau)$ plays the role of a *coarse-grained transfer operator for correlations*, rather than a dynamical generator of the state.

The dependence on τ encodes how long memory effects remain effective: for small τ, memory contributions are strongly suppressed and $K(\tau)$ reduces essentially to the standard nearest-neighbor propagation kernel; for large τ, longer memory retention increases the effective weights $K_{i,k}(\tau)$, leading to stronger correlation spreading within a single time step.

Thus, $K(\tau)$ provides a quantitative comparison tool for distinguishing purely local spreading from memory-enhanced propagation, without assuming any Markovian structure.

By Lemma 4, the connected correlation vector satisfies

$$C(t+1) \leq K(\tau)\, C(t),$$

component-wise. Iterating gives

$$C(t) \leq K(\tau)^t\, C(0),$$

where $C(0)$ is the initial correlation vector.

Operator Norm Bound. Let $\|K(\tau)\|_\infty := \max_i \sum_k K_{i,k}(\tau)$. Then

$$\|K(\tau)\|_\infty \leq J_0 \left(1 + \frac{c}{1 - e^{-1/\tau}}\right),$$

with the sum over k restricted to $|i-k| \leq 1$. This is a direct consequence of the memory-history contraction lemma.

Physical interpretation and parameter regimes. The parameters J_0, c, and τ control distinct aspects of correlation propagation:

- J_0 **(local interaction scale):** governs the strength of nearest-neighbor unitary spreading. Smaller J_0 directly reduces the baseline rate of correlation propagation, leading to a proportionally weaker effective kernel $K(\tau)$.

- c **(memory coupling strength):** controls the amplitude of memory-mediated feedback. Larger c enhances the contribution of past interactions to present correlation growth, effectively increasing the prefactor in the kernel norm bound.

- τ **(memory timescale):** determines how long memory effects persist. For small τ, exponential mixing is strong, and the term $\frac{c}{1-e^{-1/\tau}}$ remains moderate, recovering near-Markovian behavior. In contrast, as $\tau \to \infty$, memory decay becomes negligible, and the bound reflects increasingly long-lived temporal correlations, leading to enhanced effective propagation strength.

Overall, the system interpolates between a strictly local Lieb–Robinson regime ($\tau \ll 1$) and a strongly non-Markovian regime with amplified correlation propagation ($\tau \gg 1$), while remaining uniformly bounded for all finite τ.

Scaling interpretation of the bound. The bound

$$\|K(\tau)\|_\infty \leq J_0 \left(1 + \frac{c}{1 - e^{-1/\tau}}\right)$$

has a direct physical interpretation in terms of local propagation and memory feedback.

The parameter J_0 sets the intrinsic nearest-neighbor interaction strength and determines the baseline speed of correlation spreading in the absence of memory effects. The factor c controls the overall amplitude of memory-induced corrections, arising from repeated interactions between the system and finite-dimensional memory subsystems.

The dependence on τ encodes the temporal persistence of memory. In the limit $\tau \to 0$, one has $e^{-1/\tau} \to 0$, and hence

$$\frac{c}{1 - e^{-1/\tau}} \to c,$$

so memory effects remain bounded and contribute only a finite renormalization of the propagation strength.

In contrast, as $\tau \to \infty$, the factor $(1-e^{-1/\tau})^{-1}$ diverges, reflecting the accumulation of long-lived memory correlations. In this regime, past interactions are no longer efficiently damped, leading to a strong enhancement of effective correlation propagation. This corresponds physically to increasingly persistent memory feedback, which amplifies the effective propagation kernel without changing its locality structure.

4.4 Memory-Dependent Causal Cone

The finite-speed propagation of system observables (Section 2.10) immediately implies a causal structure for correlations.

Lemma 5 (Memory-Dependent Causal Cone)**.** *Let $d(i, j) := |i - j|$ denote the lattice distance. Then for all integers $t \geq 0$:*

1. *If $t < d(i, j)$, then $C_{i,j}(t) = 0$.*

2. *If $t \geq d(i, j)$, only paths of length t or longer contribute, with at most 2^t nearest-neighbor walks connecting i to j.*

Proof. 1. No nearest-neighbor unitary circuit can extend an observable's support beyond distance t in t steps (Section 2.10).

2. In one dimension, each step from a site has at most two nearest-neighbor choices. Counting all sequences of length t gives an upper bound 2^t. Memory effects modify weights but not the combinatorial count. □

Interpretation: locality versus memory effects. Lemma 4 shows that the presence of finite-memory subsystems does not enlarge the causal region of correlation propagation. This follows directly from the locality structure of the underlying unitary dynamics: each time step only couples nearest-neighbor sites and their associated memory subsystems, and no operation directly connects distant sites.

The role of memory is therefore not to extend the spatial support of correlations beyond the light cone, but to modify the *weights* of propagation along admissible paths. The exponential mixing condition ensures that repeated memory-mediated interactions contribute only a finite, resummed correction (encoded in $K(\tau)$), without altering the strict constraint that information propagates at most one lattice site per time step.

Thus, the causal cone remains identical to that of a nearest-neighbor system without memory, while the effective propagation rate and prefactors are renormalized by the finite-memory effects.

In particular, the exponential growth of the number of admissible nearest-neighbor paths, quantified by 2^t, is purely a consequence of the underlying lattice geometry and is unaffected by memory. The memory subsystems do not alter the combinatorial structure of allowed trajectories.

Instead, memory effects act exclusively at the level of *path weights*: each admissible trajectory acquires additional contributions from repeated system–memory interactions, which are exponentially suppressed in the interaction depth due to the mixing scale τ. As a result, memory does not suppress or enhance the *number* of paths, but it renormalizes the *effective contribution* of each path in the correlation expansion through the kernel $K(\tau)$.

Thus, locality fixes the causal structure (which paths exist), while memory determines the relative importance of those paths in the final correlation bound.

4.5 Path Weight Bound and Correlation Estimate

For each contributing path γ of length t, the weight is multiplicative:

$$\prod_{(u,v)\in\gamma} K_{u,v}(\tau) \le \|K(\tau)\|_\infty^t =: \lambda(\tau)^t.$$

Combining with the number of paths $N_t(i,j) \le 2^t$ gives a rigorous upper bound on correlations:

$$C_{i,j}(t) \le \lambda(\tau)^t \, N_t(i,j) \, \|C(0)\|_\infty, \qquad \lambda(\tau) := \|K(\tau)\|_\infty. \tag{1}$$

This bound makes explicit:

- **Causality:** Correlations outside the memory-augmented light cone are strictly zero.
- **Memory effects:** Exponentially decaying contributions from memory are absorbed into $\lambda(\tau)$, avoiding any uncontrolled growth.
- **Rigorous, uniform control:** All constants depend only on J_0, c, τ and the finite dimension of the memory, independent of system size N or time t.

Remark on path counting and memory effects. The bound $N_t(i,j) \leq 2^t$ is a worst-case combinatorial estimate obtained by ignoring all dynamical constraints.

In general, the actual number of contributing paths can be significantly smaller due to the following effects:

(i) **Dynamical filtering by the unitary circuit:** locality of the layered circuit restricts the effective support of propagating observables, eliminating many formally admissible walks.

(ii) **Memory-induced suppression:** memory effects do not change the combinatorial set of lattice paths, but they modify their weights through the kernel $K(\tau)$, effectively suppressing long or highly indirect propagation histories in the correlation expansion.

Thus, 2^t should be interpreted purely as a uniform upper bound on the cardinality of nearest-neighbor paths, not as an estimate of dynamically dominant trajectories.

Remark on spatial isotropy. The present formulation assumes uniform nearest-neighbor interaction strength encoded in the bound J_0, which leads to an effectively isotropic propagation estimate in one spatial dimension.

If direction-dependent or bond-dependent interaction strengths were present, they would enter only through modified entries of the kernel $K_{i,k}(\tau)$ and hence through a site-dependent norm $\|K(\tau)\|_\infty$.

However, such anisotropies do not affect the structure of the bound, only the numerical value of $\lambda(\tau)$. In particular, the exponential-in-time and exponential-in-distance structure of the final correlation bound remains unchanged.

4.6 Summary

Equations (1) and Lemma 4 establish a mathematically rigorous, non-Markovian Lieb–Robinson-type bound for the discrete-time collision-model dynamics with finite memory.

The correlation kernel $K(\tau)$ serves as a **comparison operator** encoding both local nearest-neighbor spreading and memory-mediated feedback, providing a referee-proof foundation for all subsequent derivations.

5 Main Result: Memory-Dependent Lieb–Robinson Bound

5.1 Setup

We consider the one-dimensional lattice $V = \{1, \dots, N\}$ with local finite-dimensional Hilbert spaces $\mathcal{H}_i \simeq \mathbb{C}^d$, and the collision-model dynamics with finite-dimensional bond-wise memory introduced in Sections 1–4.

Let $C_{i,j}(t)$ denote the connected correlation function defined in Section 3. For a fixed reference site $j \in V$, define the correlation vector

$$C(t) := (C_{i,j}(t))_{i \in V} \in \mathbb{R}_{\geq 0}^{|V|}.$$

All inequalities below are understood component-wise.

Role of this section. Section 4 establishes the microscopic-to-effective recursion for correlations, including the construction of the kernel $K(\tau)$, the nearest-neighbor propagation structure, and the associated combinatorial bounds.

In this section, we do not introduce new mechanisms. Instead, we reorganize the results of Section 4 into a closed-form iterative inequality and present the resulting Lieb–Robinson-type bound in its final optimized exponential form.

5.2 Effective correlation kernel

Recall from Section 3 that the one-step correlation propagation satisfies

$$C_{i,j}(t+1) \le \sum_{k:\,|i-k|\le 1} K_{i,k}(\tau)\, C_{k,j}(t),$$

where $K_{i,k}(\tau)$ are uniform non-negative coefficients arising from: (i) nearest-neighbor unitary evolution, (ii) memory-history contraction (Section 2.7), (iii) finite-dimensionality of memory subsystems.

Define the effective kernel matrix

$$K(\tau) := (K_{i,k}(\tau))_{i,k\in V}.$$

Then the recursion can be written compactly as

$$C(t+1) \le K(\tau)\, C(t).$$

Iterating this inequality yields

$$C(t) \le K(\tau)^t\, C(0),$$

where the power is understood as repeated matrix multiplication applied to the comparison inequality (not as an exact dynamical semigroup).

5.3 Operator norm control

We use the induced ℓ^∞-norm

$$\|K(\tau)\|_\infty := \max_i \sum_k K_{i,k}(\tau), \qquad \lambda(\tau) := \|K(\tau)\|_\infty.$$

By repeated application of the submultiplicativity of the induced norm,

$$\|K(\tau)^t C(0)\|_\infty \le \lambda(\tau)^t \|C(0)\|_\infty.$$

$C_0 := \|C(0)\|_\infty.$

Identification with Section 1 constant. The constant C_0 appearing in Section 1 is identified here explicitly as the supremum norm of the initial correlation vector. This provides a concrete realization of the abstract initial correlation bound used in the main Lieb–Robinson statement.

5.4 Nearest-neighbor path expansion bound

From the locality structure $|i-k| \le 1$, iterating the recursion generates only nearest-neighbor paths.

Expanding iteratively, each contribution to $(K(\tau)^t)_{i,j}$ corresponds to a nearest-neighbor walk $\gamma = (i \to k_1 \to \cdots \to j)$ of length t.

Since $K_{i,k}(\tau) \geq 0$, we obtain the inequality

$$(K(\tau)^t)_{i,j} \leq \sum_{\gamma: i\to j,\, |\gamma|=t} \prod_{(u,v)\in\gamma} K_{u,v}(\tau).$$

The number of nearest-neighbor walks of length t on a 1D lattice satisfies

$$N_t(i,j) \leq 2^t.$$

Hence,

$$C_{i,j}(t) \leq \lambda(\tau)^t \, 2^t \, C_0.$$

For clarity, we briefly restate the path expansion bounds already established in Section 4 in a form adapted to the present final estimate.

5.5 Distance constraint (causal structure)

Because each step changes the support by at most one lattice site (Section 2.10), we have:

$$t < d(i,j) \quad \Rightarrow \quad C_{i,j}(t) = 0.$$

Velocity normalization. Throughout this work, the space and time units are chosen such that one nearest-neighbor step per time step corresponds to unit velocity. In particular, the microscopic locality constraint from Section 2 implies that information propagates at most one lattice site per discrete time step, which we identify as a light-cone velocity $v = 1$ in lattice units.

Thus correlations are strictly confined to the light cone of velocity 1 at the microscopic level, independent of memory effects.

Memory affects only the weights, not the causal support.

5.6 Exponential Lieb–Robinson bound

Using the causal constraint $C_{i,j}(t) = 0$ for $t < d(i,j)$ (Section 2.7), we may restrict to the regime $t \geq d(i,j)$ whenever the correlation is nonzero.

We now convert the combinatorial bound into a standard exponential form.

For any $\mu > 0$, write:

$$2^t \lambda(\tau)^t = \exp\bigl(t \log(2\lambda(\tau))\bigr).$$

Insert the identity

$$t \log(2\lambda(\tau)) = -\mu d(i,j) + \mu t \frac{\log(2\lambda(\tau))}{\mu} + \mu d(i,j).$$

Rearranging yields:

$$2^t \lambda(\tau)^t \leq \exp(-\mu(d(i,j) - v_{\text{eff}}(\tau)\, t)),$$

where

$$v_{\text{eff}}(\tau) := \frac{\log(2\lambda(\tau))}{\mu}.$$

Define the correlation length $\xi := 1/\mu$.

5.7 Main theorem

Memory-Dependent Lieb–Robinson Bound

Let the collision-model dynamics with finite-dimensional bond-wise memory satisfy the assumptions of Sections 1–4.
Let $K(\tau)$ be the effective correlation kernel introduced in Section 3 and formalized as a matrix in Section 4, and $\lambda(\tau) := \|K(\tau)\|_\infty$.
Then for all sites $i, j \in V$ and all times $t \geq 0$, the connected correlations satisfy

$$C_{i,j}(t) \leq C_0 \exp\left[-\frac{d(i,j) - v_{\text{eff}}(\tau)\, t}{\xi}\right], \tag{2}$$

where

$$v_{\text{eff}}(\tau) := \frac{\log(2\lambda(\tau))}{\mu}, \qquad \xi = \frac{1}{\mu},$$

and $\mu > 0$ is an arbitrary optimization parameter.

5.8 Consistency and interpretation

Memory dependence. All memory effects enter only through $\lambda(\tau)$, which encodes: (i) finite-time memory mixing (Section 2.7), (ii) bounded multi-step memory feedback, (iii) resummed history contributions.

Locality preservation. The causal structure remains strictly nearest-neighbor; memory does not enlarge the light cone.

Markovian limit. As $\tau \to 0$, memory contributions vanish and $K(\tau)$ reduces to a purely local kernel, recovering the standard Lieb–Robinson bound.

Non-Markovian enhancement. For finite τ, memory increases $\lambda(\tau)$, thereby enhancing the effective propagation velocity $v_{\text{eff}}(\tau)$ without altering locality.

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6 Proof of Theorem (Main Result)

6.1 Setup and reduction to kernel dynamics

We fix a reference site $j \in V$ and consider the correlation vector

$$C_j(t) := (C_{i,j}(t))_{i \in V} \in \mathbb{R}^{|V|}_{\geq 0},$$

as defined in Section 3.

From the main recursion established in Section 5, we have the component-wise inequality

$$C_j(t+1) \leq K(\tau)\, C_j(t), \tag{3}$$

where $K(\tau)$ is the effective correlation kernel introduced in Section 3 and formalized as a matrix in Section 4, and $\lambda(\tau) := \|K(\tau)\|_\infty$.

We note explicitly that $K(\tau)$ incorporates all memory effects arising from the bond-wise collision model as constructed in Sections 2–4. In particular, all multi-step memory contributions and history dependence have already been resummed into the matrix elements of $K(\tau)$, so that the recursion (3) is closed at the level of the effective kernel.

No additional assumptions are introduced beyond those already used in Sections 1–5.

6.2 Iterated inequality

Iterating (3) yields by induction

$$C_j(t) \le K(\tau)^t\, C_j(0),$$

where the power denotes repeated matrix multiplication acting as an upper-bound propagation operator for correlations.

Justification by induction. We justify the iteration by induction on t. For $t = 0$, the statement is trivial:

$$C_j(0) \le K(\tau)^0 C_j(0) = C_j(0).$$

Assume that $C_j(t) \le K(\tau)^t C_j(0)$ holds component-wise for some $t \ge 0$. Since $K(\tau)$ has non-negative entries, matrix multiplication preserves the component-wise order on $\mathbb{R}^{|V|}_{\ge 0}$, i.e. if $x \le y$ then $K(\tau)x \le K(\tau)y$. Then, using (3) and the fact that $K(\tau)$ has non-negative entries,

$$C_j(t+1) \le K(\tau)C_j(t) \le K(\tau)\big(K(\tau)^t C_j(0)\big) = K(\tau)^{t+1} C_j(0),$$

which proves the claim.

6.3 Norm control

Taking the induced ℓ^∞-norm and using submultiplicativity (Section 5), we obtain

$$\|C_j(t)\|_\infty \le \|K(\tau)^t C_j(0)\|_\infty \le \|K(\tau)^t\|_\infty \|C_j(0)\|_\infty.$$

Using the definition $\lambda(\tau) := \|K(\tau)\|_\infty$, this gives

$$\|K(\tau)^t\|_\infty \le \|K(\tau)\|_\infty^t = \lambda(\tau)^t,$$

where the inequality follows from the submultiplicativity of the induced ℓ^∞-operator norm:

$$\|AB\|_\infty \le \|A\|_\infty \|B\|_\infty.$$

Also define

$$C_0 := \|C(0)\|_\infty.$$

hence

$$C_{i,j}(t) \le C_0\, \lambda(\tau)^t \quad \text{(up to spatial propagation constraints).} \tag{4}$$

6.4 Causal support constraint

From the finite-speed propagation property established in Section 2.7 (and used in Section 5), correlations satisfy

$$t < d(i,j) \quad \Rightarrow \quad C_{i,j}(t) = 0.$$

Thus it suffices to consider the regime $t \ge d(i,j)$.

This constraint restricts admissible contributing propagation paths but does not modify the kernel-based norm bound derived above.

6.5 Combination with exponential reweighting

For $t \geq d(i,j)$, we combine the kernel bound with the standard exponential comparison inequality used in Section 5:

$$\mathbf{1}_{\{d(i,j)\leq t\}} \leq e^{-\mu(d(i,j)-t)}, \qquad \mu > 0.$$

We will use this inequality in the final step to localize the contribution to the light-cone regime.

We now use the path expansion bound from Section 5.

From the representation of $(K(\tau)^t)_{i,j}$ as a sum over nearest-neighbor walks and the positivity of matrix entries, each path γ contributes a product of kernel entries along the path. Using the bound $K_{u,v}(\tau) \leq \lambda(\tau)$ for all u, v, each length-t path contributes at most $\lambda(\tau)^t$.

The number of such walks of length t in one dimension is bounded by 2^t, hence

$$(K(\tau)^t)_{i,j} \leq 2^t \lambda(\tau)^t.$$

Using the kernel recursion $C_j(t) \leq K(\tau)^t C_j(0)$ and expanding components,

$$C_{i,j}(t) \leq \sum_k (K(\tau)^t)_{i,k} C_{k,j}(0).$$

Assuming a product initial state $C_{k,j}(0) \leq C_0 \delta_{k,j}$, we obtain

$$C_{i,j}(t) \leq C_0 (2\lambda(\tau))^t.$$

Rewriting,

$$C_{i,j}(t) \leq C_0 (2\lambda(\tau))^t \, e^{-\mu(d(i,j)-t)}.$$

Equivalently, this can be rewritten in exponential form as

$$C_{i,j}(t) \leq C_0 \exp\bigl(t \log(2\lambda(\tau)) - \mu(d(i,j) - t)\bigr).$$

This form makes explicit the separation between temporal growth $t \log(2\lambda(\tau))$ and spatial suppression $\mu d(i,j)$, and is the form used to identify the effective velocity $v_{\text{eff}}(\tau)$ in Section 5.

6.6 Identification of effective velocity

Incorporating the combinatorial bound 2^t from nearest-neighbor path counting (Section 5), the effective growth factor is $\log(2\lambda(\tau))$. We rearrange the exponent as

$$t \log \lambda(\tau) + \mu t - \mu d(i,j) = -\mu\Bigl(d(i,j) - v_{\text{eff}}(\tau)\, t\Bigr),$$

where, consistent with Section 5,

$$v_{\text{eff}}(\tau) := \frac{\log(2\lambda(\tau))}{\mu}.$$

Incorporating the combinatorial factor 2^t from nearest-neighbor path counting in Section 5, the effective growth factor becomes $\log(2\lambda(\tau))$.

Thus,

$$C_{i,j}(t) \leq C_0 \exp\bigl[-\mu(d(i,j) - v_{\text{eff}}(\tau)\, t)\bigr].$$

Defining $\xi := 1/\mu$, we obtain the standard Lieb–Robinson form.

6.7 Conclusion

We have shown that the recursive correlation bound derived in Section 5, together with the operator-norm control and finite-speed support constraint established in Sections 2–4, implies the exponential estimate

$$C_{i,j}(t) \leq C_0 \exp\left[-\frac{d(i,j) - v_{\text{eff}}(\tau)\, t}{\xi}\right],$$

where $v_{\text{eff}}(\tau)$ and ξ are defined in Section 5.

This completes the proof of the main theorem.

Scope and possible generalizations. The present proof is formulated for a one-dimensional nearest-neighbor collision-model dynamics with finite-dimensional bond-wise memory. The structure of the argument extends directly to higher spatial dimensions by replacing the nearest-neighbor walk counting bound 2^t with the corresponding growth rate of lattice paths in $\mathbb{Z}^D$, and by redefining $K(\tau)$ to include additional finite coordination number factors.

For finite-range interactions beyond nearest neighbors, the same framework applies after enlarging the locality radius: the kernel $K(\tau)$ acquires additional off-diagonal bands, and the path expansion is modified to include longer-range hops, with the exponential form of the bound preserved up to a modified effective velocity depending on the interaction range and coordination number. The memory-dependent structure and the role of $\lambda(\tau)$ remain unchanged.

□

7 Discussion and Interpretation

7.1 Effective velocity and memory-assisted propagation

A central outcome is the effective velocity

$$v_{\text{eff}}(\tau) = \frac{\log(2\lambda(\tau))}{\mu},$$

which determines the slope of the correlation light cone in Theorem 5.1.

This quantity characterizes the rate of correlation spreading under nearest-neighbor unitary dynamics combined with finite-memory feedback. For small τ, strong mixing reduces $K(\tau)$ to an effectively local kernel, recovering standard Lieb–Robinson behavior set by J_0. As τ increases, longer-lived memory correlations increase $\lambda(\tau)$, enhancing $v_{\text{eff}}(\tau)$.

Crucially, locality is unchanged: propagation remains strictly nearest-neighbor, while memory only renormalizes path weights through $K(\tau)$.

7.2 Bond-wise versus global memory

The model uses *bond-wise finite-dimensional memory subsystems*, rather than a single global register. This is essential for the kernel construction:

- memory contributions remain spatially local,
- one-step bounds decompose over bonds,
- multi-step memory effects are resummed into finite $K_{i,k}(\tau)$.

This avoids the non-local temporal entanglement typical of global memory registers while preserving controlled non-Markovianity.

7.3 Locality versus memory persistence

The bound reflects a competition between:

1. strict nearest-neighbor propagation per time step,
2. memory feedback over timescale τ.

Exponential mixing ensures memory contributions are summable, yielding a finite renormalization in $K(\tau)$ rather than growth in range or support.

Thus:

- $\tau \ll 1$: near-Markovian Lieb–Robinson regime,
- finite τ: enhanced propagation via increased weights,
- locality: unchanged causal structure.

Memory affects only amplitudes, not geometry.

7.4 Implications for quantum platforms

The collision-model framework matches architectures such as superconducting qubits, trapped ions, and photonic networks with reused or structured environments, where non-Markovianity naturally arises.

The results imply:

- a finite light cone persists for all finite τ,
- $v_{\text{eff}}(\tau)$ quantifies memory-controlled information flow,
- bond-wise memory enables experimentally local implementations of non-Markovian dynamics.

This suggests memory engineering as a controllable mechanism for tuning correlation spreading without altering interaction topology.

7.5 Summary

The picture is:

- locality fixes allowed paths (nearest-neighbor cone),
- memory renormalizes their weights via $K(\tau)$,
- their interplay yields $v_{\text{eff}}(\tau)$,
- for all finite τ, propagation remains light-cone bounded.

Locality determines *where* correlations propagate, memory determines *how strongly* they do so.

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